PROBABILITY-WEIGHTED DISCOUNT RATES IN CAPITAL BUDGETING

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Abstract. The typical present value function (of a cash flow, as a function of increasing discount rates) is nonlinear, a familiar fact well known to finance practioners. It's not difficult to overlook, however, the effect this nonlinearity imposes on the very intuitive concept of determining expected discount rates as probability-weighted averages, particularly when the two concepts commingle in a capital-budgeting context. A bit of foresight can help avoid an erroneous calculation result.

Introduction and base example

It's probably a fair assessment that this one belongs in the "*Not Exactly Groundbreaking But Potentially Useful*" file—assuming you maintain such a thing. While helping a student the other day with a fairly routine capital budgeting problem, I was reminded that in situations involving **probability-weighted discount rate estimations**, one's intuitive approach might indeed lead one astray.

A rather stylized and over-simplified illustration provides the motivation, and sets the stage for a more generalized approach. Suppose for example you know you'll have an opportunity to drop some coin into a project one year hence. This project will generate a single cash flow of 100,000[1] five years (i.e., at $t = 6$) after the initial investment. Certainly one of your first moves is an estimation of this project's present value, perhaps to decide if further investigation is warranted.

Certainly you have no precise knowledge of what the appropriate 5-year discount rate will be one year hence, when it's time to pull the trigger on this deal. However, you've been provided with an estimation that the best discount rate[2] will either be $r_1 = 10\%$ or $r_2 = 20\%$, each with a probability of 1/2. You assign to your junior analyst Simeon the task of pricing this project, given the available data. Having done so, you then recall that Simeon has exhibited questionable judgment in the past, so just for safe measure you assign this same task to Dori—a more seasoned senior analyst—to provide (hopefully) an independent confirmation of the results.

Simeon reasons that the expected discount rate $E(r)$ is

$$E(r) = \frac{1}{2} \cdot r_1 + \frac{1}{2} \cdot r_2 = 15\%$$

He then proceeds to PV the project (as of $t = 1$) ...

$$100,000\,[1 + E(r)]^{-5} = 100,000(1.15)^{-5} = \mathbf{49,718}$$

Dori on the other hand uses a different bit of reasoning in her approach. She determines the project's PV under each of the discount-rate possibilities, and *then* applies the weights:

$$\left(\frac{1}{2}\right)\left[\frac{100,000}{(1+r_1)^5}\right] + \left(\frac{1}{2}\right)\left[\frac{100,000}{(1+r_2)^5}\right] = \cdots = \mathbf{51,140}$$

Oops. While the magnitude of the difference is barely a rounding error here, the bigger point is that the two approaches—either one of which might reasonably be seen as logical—yield different results. In a different context (large number of cash flows? longer project life or duration? asymmetric probabilities?) the spread between the different results might indeed become significant.

What's disconcerting is that, upon reflection, the logic used by *both* analysts seems on its face to have merit, at least intuitively. Simeon prices the single cash flow using an *expected discount rate*, whereas Dori hits the set of possible values with the corresponding probability weights to derive an *expected* PV. C'mon, admit it; until you give it some thought it's not immediately clear who's got it right. Seems we'll have to put it under a bright light and examine the inner workings.

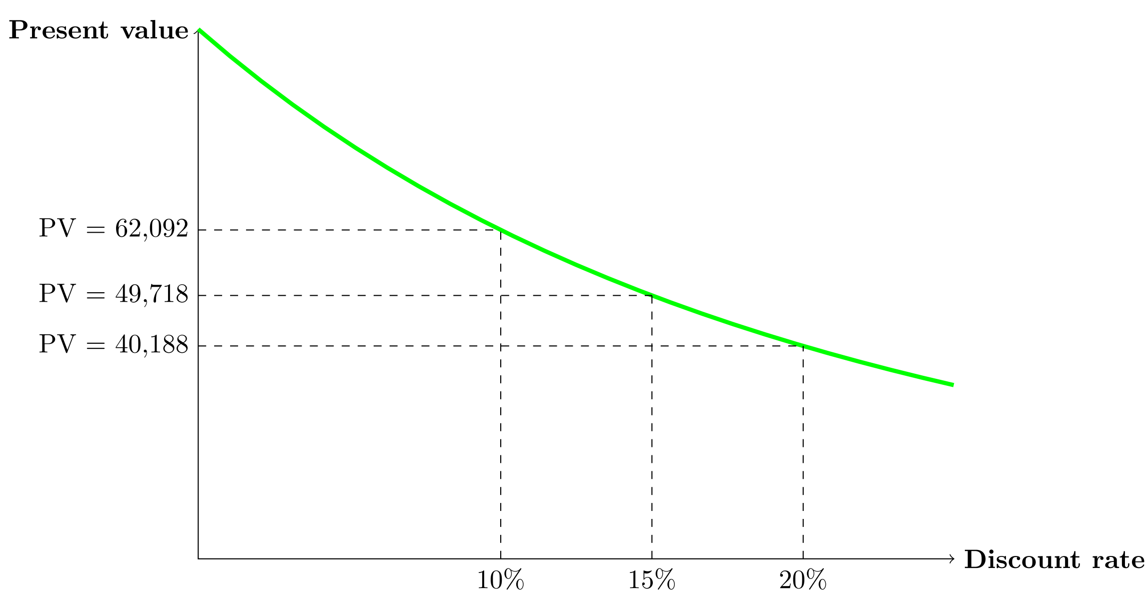
What went wrong, and how to fix it

(Spoiler alert) Dori nailed it. To see this rather easily, assume that two identical projects are undertaken simultaneously. Suppose further—while suspending economic theory for a moment—that the two projects have appropriate discount rates of 10% and 20%, respectively. Keeping with the foregoing illustration, each project throws off a single cash flow of 100K, with a 5-year maturity. Then the aggregate value of the two projects together is

$$100,000(1.1)^{-5} + 100,000(1.2)^{-5} = 102,280$$

for a mean PV of **51,140** per. In fact, it's easy to see how Dori's model follows the same mathematical process, generally speaking.

So where did Simeon's approach come off the rails? The answer rests with the implicit *linearity* of his approach, which doesn't cut it very well in the decidedly *nonlinear* world of discounting. Simply put, while 15% is indeed the simple weighted average of the two possibilities 10% and 20%, the average value of two projects, priced at 10% and 20%, respectively, is **not** equivalent to a single project discounted at 15%. Putting it a bit differently, experiencing an actual discount rate of 10% half the time and 20% half the time, isn't at all the same as experiencing an actual discount rate of 15% **all** the time, *ceteris paribus*. A simple graph showing PV of a single cash flow (amount = 100,000), as a function of the discount rate, illustrates it in the following figure.



In the figure, the (horizontal) symmetry between two discount rates (10% and 20%) and their simple mean is contrasted against the (vertical) nonsymmetrical relationship among their corresponding present values (courtesy of the hyperbolic function)[3].

The unrealistic setup used as the backdrop for the discussion thus far is too simplified to have much direct real-world relevance. But I'll leave it to the reader to expand on the idea (should anyone so desire; a questionable proposition, perhaps) and generalize to more realistic scenarios involving multiple cash flows, asymmetry across the probability weights, and other variants.

Still, the point's been made: **In a capital budgeting (or similar) application, when pricing cash flows using a set of possible discount rates—as opposed to a single assumed rate—be sure the corresponding probability weights are used to develop an *expected* present value from the individual state-specific PVs, rather than to develop an *expected* discount rate which is then used to price the cash flows**.

That said, though, is there a way to salvage Simeon's method, preserving the appealing logic of his approach while also producing a correct result? Yes, indeed, if we employ the probability weights not to derive an expected discount *rate*, but rather an expected discount *factor*. In other words, let's use the discount rates in their *rational* form. When Simeon revises his computations in this manner (on his lunch hour, natch) his discount factor becomes ...

$$(1 + r_1)^{-5} \left(\frac{1}{2}\right) + (1 + r_2)^{-5} \left(\frac{1}{2}\right) = \cdots = 0.5114$$

His pricing computation then results in

$$100,000 \times 0.5114 = 51,140$$

Much better. By applying the weights to the discount rates in their rational form—as they're used in the discounting process—rather than to their 'percentage' form, the nonlinearity is automatically taken into account. Still, Dori got it right the first time ...

Epilogue

... which made your decision to put Dori in direct supervision of Simeon seem an easy choice. Now, having offloaded some of your supervisory duties to Dori, what better way to toast your managerial acumen than by knocking off early this afternoon?

But while traversing the office parking lot toward your car, a glimmer of an idea begins taking shape in your mind: "*Let's turn this around—does this principle hold true in reverse*?" You recall that tomorrow's to-do list includes some *future value* computations the CFO ordered up to support the board's evaluation of some expansion opportunities. "*Will the computations be swayed to different results, depending on whether we apply the weights to the yields, rather than to the set of possible cash flow outcomes*?"

You make a mental note to think about that one—tomorrow.

An aside

At its core, the generic PV function belongs to the family of **rational** functions, and hence behaves as a *hyperbola*—more specifically, the physical constraints confine it strictly to a first quadrant range and domain. For any constant compounding period t and any given cash flow C_t, the present value of C_t as a function of the discount rate r, sets up as

$$PV(r) = \frac{C_t}{(1 + r)^t}$$

The denominator is a t^{th}-degree polynomial of r, making PV a card-carrying member of the rational-function fraternity.

Notes

[1]To reduce clutter I tend to favor omission of currency symbols in these generic situations where we're dealing with units of any arbitrary currency.

[2]The annualized 5-year spot rate expected to be in effect at $t = 1$.

[3]See the Aside, above.